THE STANDARD MODEL AND THE LATTICE

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I discuss some of the difficulties with formulating chiral symmetry on the lattice and review a recently proposed scheme for a fully finite and exactly gauge invariant lattice regularization of the standard model.

1 Introduction

Lattice gauge theory, now in its third decade, has since its inception been plagued by difficulties with fermions. There are two apparently distinct fermion problems. First is the issue of algorithms, arising since fermionic fields are anti-commuting variables. Since the exponentiated action is an operator in a Grassmann space, comparisons with random numbers for a Monte Carlo program are problematical. Several ways around this have been devised, mostly based on integrating the fermions analytically as a determinant, but in my opinion the approaches remain awkward. Furthermore, when there is a background baryon density, *i.e.* a chemical potential term in the action, cancelations between terms of varying phase make the problem essentially unsolved except on the tiniest lattices. This is not a unique problem to lattice gauge theory; studying doping in many electron models has equivalent difficulties.

In this talk, however, I concentrate on the other fermion issue: chiral symmetry and doubling. Since the lattice is a first principles approach to field theory, one could ask why care about the details of chiral symmetry. Just put the problem on the computer, predict particle properties, and they should come out correctly if the underlying dynamics is relevant. While this may perhaps be a logical point of view, it ignores a vast lore built up over the years in particle physics. In the context the strong interactions, the pion and the rho mesons are made of the same quarks, the only difference being whether the spins are anti-parallel or parallel. Yet the pion, at 140 MeV, weighs substantially less than the 770 MeV rho. Chiral symmetry is at the core of the conventional explanation. Since the up and down quarks are fairly light, we have an approximately conserved axial vector current, and the pion is believed to be the remnant Goldstone boson of a spontaneous breaking of this chiral symmetry.

Another motivation for studying chiral issues on the lattice arises when

considering the weak interactions. Here we are immediately faced with the experimental observation of parity violation, neutrinos are left handed. In the standard electroweak model, fundamental gauge fields are coupled directly to chiral currents. The corresponding symmetries are gauged, *i.e.* they become local, and are crucial to the basic structure of the theory. Since the lattice is the one truly non-perturbative regulator for defining a field theory, if one cannot find a lattice regularization for the standard model, the standard model itself may not be well defined.

Regulating divergences via a lattice is by no means a unique process. However, Wilson's original formulation [1] has some rather remarkable properties when applied to strong quark confining dynamics, usually called QCD. First, the approach is indeed a regulator: it makes the theory fully finite. Second, the cutoff is non-perturbative, unlike more conventional approaches which begin by finding a formally divergent Feynman diagram and then cutting it off. But diagrams are the basis of perturbation theory. The advantage of the lattice is its imposition before any expansion. Third, and perhaps the most remarkable, the Wilson approach accomplishes the above two feats while maintaining an exact local gauge symmetry. Besides its inherent elegance, this precludes the need for any gauge variant counter-terms in the renormalization procedure. Since the theory is fully finite at the outset, the whole issue of gauge fixing is circumvented.

Given these features, it is natural to ask if a similar scheme exists for the full standard model, including the gauged chiral currents. The answer to this is, I believe, unknown. Nevertheless, I will lead this talk towards a scheme that may provide all of the above features. It involves some rather complex couplings, opening possible routes to failure, but does appear to include the necessary features for such a formulation.

2 What is chiral symmetry?

For pedagogy I digress briefly on what I mean by chiral symmetry. The issues here are deeply entwined with representations of the Lorentz group – zero mass particles are special. In particular, the helicity of a massless particle is invariant under Lorentz transformations. Each helicity state forms a separate representation of the Lorentz group; for spin one-half the Dirac field can be separated into right and left handed parts, ψ_R and ψ_L, formally independent. Furthermore, because of the way the Dirac matrices appear, the helicity of a fermion is naively preserved under gauge interactions. When a polarized electron at high energy scatters off of an electromagnetic field, its helicity is unchanged.

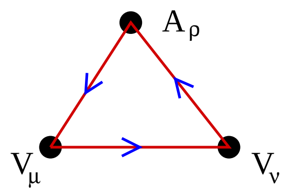

Figure 1: The triangle diagram cannot be regulated so both vector and axial vector currents are conserved.

This whole issue, however, is clouded by the so-called "chiral anomalies." In particular, the famous triangle diagram, sketched in Fig. 1, coupling two vector and one axial vector current is divergent, and no regularization can keep them both conserved. If either is coupled to a gauge field, such as the electromagnetic current to the vector current, this diagram must be regulated with that particular current being conserved. Then the other cannot be. These anomalies are at the core of the lattice problems.

The concept of chirality becomes even simpler in one spatial dimension, where the direction of motion of a massless particle is invariant under boosts. Then the anomaly is easily understood via simple band theory.[2] A particle of non-zero mass m and momentum p has energy $E = \pm\sqrt{p^2 + m^2}$. Here I use a Dirac sea description where the negative energy states are to be filled in the normal vacuum. Considering the positive and negative energy states together, the spectrum of the system has a gap equal to twice the particle mass. In the vacuum the Fermi level is at zero energy, exactly in the center of this gap. In conventional band theory language, the vacuum is an insulator.

In contrast, for massless particles where $E = \pm|p|$, the gap vanishes. The system becomes a conductor, as sketched in Fig. 2. Of course, conductors can carry currents, and here the current is proportional to the number of right moving particles minus the number of left movers. If we consider gauge fields, they can induce currents, a process under which the number of right or left movers cannot be separately invariant. This is the anomaly, without which transformers would not work.

This induction of currents is not a conversion of particles directly from left into right movers, but rather a sliding of levels in and out of the Dirac sea. The generalization of this discussion to three spatial dimensions uses Landau levels in a magnetic field; the lowest Landau level behaves exactly as the above one dimensional case.[2]

One particularly important consequence for the standard model is that baryon number is one of the anomalous charges. Indeed, 't Hooft[3] pointed

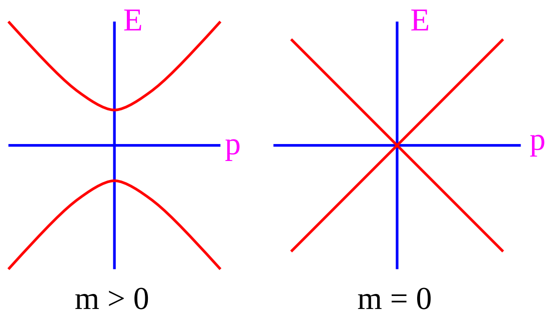

Figure 2: In one dimension the spectrum of massive particles has a gap, and the vacuum can be regarded as an insulator. The massless case, in contrast, represents a conductor. The anomaly manifests itself in the ability to induce currents in a wire.

out a specific baryon-number-changing mechanism through topologically non-trivial gauge configurations. The rate is highly suppressed due to a small tunneling factor and is far too small to observe experimentally. Nevertheless, the process is there in principle, and any valid non-perturbative formulation of the standard model must accommodate it. If we have a fully finite and exactly gauge invariant lattice theory, the dynamics must contain terms which violate baryon number. This point was emphasized some time ago by Eichten and Preskill [4] and further by Banks.[5]

Without baryon violating terms, something must fail. In naive approaches to lattice fermions the problem materializes via extra particles, the so-called doublers, which cancel the anomalies. For the strong interactions alone, a vector-like theory, Wilson [6] showed how to remove the doublers by adding a chirally non-symmetric term. This term formally vanishes in the continuum limit, but serves to give the doublers masses of order the inverse lattice spacing. As chiral symmetry is explicitly broken, the chiral limit of vanishing pion mass is only obtained with a fine tuning of the quark mass, which is no longer "protected" with the bare and physical quark masses no longer vanishing together. This approach works well for the strong interactions, but explicitly breaks a chirally coupled gauge theory. While perhaps tractable,[7] it requires an infinite number of gauge variant counter-terms to restore gauged chiral symmetries in the continuum limit. It is these features that drive us to search for a more elegant formulation.

To proceed I frame the discussion in terms of extra space-time dimensions. The idea of adding unobserved dimensions is an old one in theoretical physics, going back to Kaluza and Klein,[8] and often is quite useful in unifying different interactions. For my purposes, it allows me to separate different parts of the

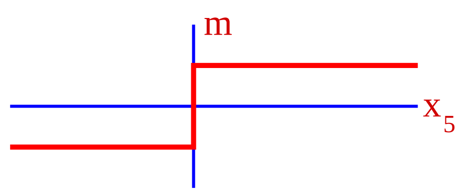

Figure 3: A step in a five dimensional fermion mass can give rise to topological zero-energy fermion modes bound to a four dimensional interface.

problem, but is probably only a theoretical crutch that can be removed at a later stage. Of course the extension of space-time to higher dimensions is crucial to modern string theories. Indeed, there are probably unexploited analogies here, in particular chiral symmetry can become quite natural when formulated on higher dimensional membranes. Here I use only the simplest extension, involving one extra dimension.

The use of an extra dimension in the context of anomalies also appears in the area of effective chiral Lagrangians. Here the famous Wess-Zumino [9] term is formulated in terms of an added fifth dimension. Anomalies in four dimensional currents are interpreted as a flow into the fifth direction. Indeed, the analogy with the following discussion is striking, and recent arguments [10] suggest a close connection between the doubling problem in lattice gauge theory and the problem of coupling gauge fields to the Wess-Zumino term.

I start with an observation of Callan and Harvey,[11] building on Jackiw and Rebbi.[12] They argue that a five dimensional massive fermion theory formulated with an interface where the fermion mass changes sign, as sketched in Fig. 3, can give rise to a four dimensional theory of massless fermionic modes bound to the interface. The low energy states on the interface are naturally chiral, and anomalous currents are elegantly described in terms of a flow into the fifth dimension.

While the Callan and Harvey discussion is set in the continuum, Kaplan [13] suggested carrying the formalism directly over to the lattice. His motivation, as mine here, is to provide a potential scheme for chiral gauge theories on the lattice. With the Wilson formulation, the particle mass is controlled via the hopping parameter, usually denoted K. The massless situation is obtained at a critical hopping, K_c, the numerical value of which depends on the gauge coupling. Thus, to set up an interface as used by Callan and Harvey, one should consider a five dimensional theory with a hopping parameter which depends on the extra fifth coordinate. This dependence should be constructed to generate a four dimensional interface separating a region with $K > K_c$ from one with $K < K_c$. Shamir [14] observed a substantial simplification on the $K < K_c$ side

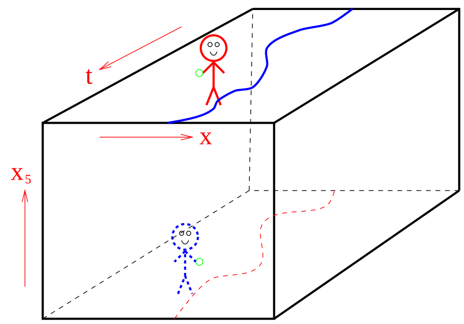

Figure 4: Regarding our four dimensional world as a surface in five dimensions.

by putting $K = 0$. Then that region decouples, and the picture reduces to a four dimensional surface of a five dimensional crystal. The physical picture is sketched in Fig. 4. For a Hamiltonian discussion, see Ref. [15]. Indeed, surface modes are not a particularly new concept; in 1939 Shockley [16] discussed their appearance in band models when the interband coupling becomes strong.

The purpose of the fifth dimension is to address fermionic issues. Since the bosonic sector of lattice gauge theory is already in good shape, I require that the gauge fields not directly see the extra coordinate. In particular, any gauge field $A(x_\mu, x_5) = A(x_\mu)$ is considered to independent of x_5. Furthermore, the gauge field has no fifth component, *i.e.* $A_5 = 0$. A possibly helpful analogy[17] regards x_5 as effectively a "flavor" index, with hopping through the extra dimension representing a somewhat complex "mixing."

This approach gives a natural chiral theory on one wall of our system. However, as Fig. 4 hints, for a finite five dimensional system there are generally additional four dimensional surfaces. One extra surface, as in the figure, I refer to as an "anti-wall." Indeed, on any finite system these topological defects occur in pairs. This raises a difficulty with the above scheme for inserting the gauge fields. Since the latter do not know about the extra dimension, they couple equally to modes on all walls. On the anti-wall there are also fermion zero modes, of the opposite chirality to those on the original wall. These are effectively "mirror" fermions; corresponding to a left handed neutrino on the original wall, a right handed neutrino appears on the anti-wall. These mirrors cannot be neglected since they couple with equal strength to the gauge fields.

Furman and Shamir [18] have argued that for vector-like theories, such as the usual strong interactions, such a formulation could be of considerable practical value. In this case non-anomalous chiral symmetries, responsible for the

lightness of the pion, would be natural in the limit of a large fifth dimension. Indeed, the doubling appearing with the anti-walls is in some sense the minimal required by famous no-go theorems [19] Preliminary efforts with this scheme have been promising.[20]

Here, however, I am interested in chiral gauge theories such as the standard model. One might imagine eliminating the extra walls by moving them off to infinity. This lies at the heart of the closely related "overlap" formalism of Ref. [17] and provides a non-perturbative definition for the chiral determinant. However, how anomaly cancelation works in this formulation is somewhat hidden, with baryon non-conservation being relegated off to infinity. Because of this, in a recent paper [21] we pursued an alternative scheme for eliminating the doublers on the anti-wall. We argued that a large four-fermion coupling on the anti-wall could induce a mass gap of order the lattice spacing for the spectrum on the "bad" wall. The form of the interaction has the same structure as the " 't Hooft" vertices responsible for the baryon non-conservation discussed above plus similar terms to break the anomalous strong axial $U(1)$ symmetry. Physically, we give the mirror protons mass by mixing them with the mirror positrons. The primary danger is that the four fermion interaction might spontaneously break one of the gauge symmetries. Such a breaking would naturally be at the scale of the lattice spacing and would destroy the model.

Rather than describing the contents of that paper in more detail, I now pursue an alternative but equivalent picture. I am motivated by the desire to understand how the no-go theorems are avoided; in particular I discuss how all basic particles can be paired so every left handed particle has a right handed counterpart. This approach, also discussed in my contribution to the Lattice '97 conference,[22] involves a rather unusual reinterpretation of the standard model.

The standard model of elementary particle interactions is based on the product of three gauge groups, $SU(3) \times SU(2) \times U(1)_{em}$. Here the $SU(3)$ represents the strong interactions of quarks and gluons, the $U(1)_{em}$ corresponds to electromagnetism, and the $SU(2)$ gives rise to the weak interactions. I ignore here the technical details of electroweak mixing. The full model is, of course, parity violating, as necessary to describe observed helicities in beta decay. This violation is normally considered to lie in the $SU(2)$ of the weak interactions, with both the $SU(3)$ and $U(1)_{em}$ being parity conserving. However, this is actually a convention, adopted primarily because the weak interactions are small. I argue below that a reassignment of degrees of freedom allows an interpretation where the $SU(2)$ gauge interaction is vector-like. Since the full model is parity violating, I must shift the parity violation into the strong, electromagnetic, and Higgs interactions.

With a vector-like weak interaction, the chiral issues move to the other gauge groups. Requiring gauge invariance for the re-expressed electromagnetism then clarifies the mechanism behind our above mentioned proposal for a lattice regularization of the standard model.[21]

To see how this works, consider only the first generation, involving four left handed doublets. These correspond to the neutrino/electron lepton pair plus three colors for the up/down quarks

$$\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L, \begin{pmatrix} u^r \\ d^r \end{pmatrix}_L, \begin{pmatrix} u^g \\ d^g \end{pmatrix}_L, \begin{pmatrix} u^b \\ d^b \end{pmatrix}_L \tag{1}$$

Here the superscripts from the set $\{r, g, b\}$ represent the internal $SU(3)$ index of the strong gauge group, and the subscript L indicates left-handed helicities.

If I ignore the strong and electromagnetic interactions, leaving only the weak $SU(2)$, each of these four doublets is equivalent and independent. I now arbitrarily pick two of them and do a charge conjugation operation, thus working with their anti-particles

$$\begin{array}{c} \begin{pmatrix} u^g \\ d^g \end{pmatrix}_L \longrightarrow \begin{pmatrix} \overline{d^g} \\ \overline{u^g} \end{pmatrix}_R \\ \begin{pmatrix} u^b \\ d^b \end{pmatrix}_L \longrightarrow \begin{pmatrix} \overline{d^b} \\ \overline{u^b} \end{pmatrix}_R \end{array} \tag{2}$$

In four dimensions anti-fermions have the opposite helicity; so, I label these new doublets with R representing right handedness.

With two left and two right handed doublets, I combine them into Dirac doublets

$$\begin{pmatrix} \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L \\ \begin{pmatrix} \overline{d^g} \\ \overline{u^g} \end{pmatrix}_R \end{pmatrix} \qquad \begin{pmatrix} \begin{pmatrix} u^r \\ d^r \end{pmatrix}_L \\ \begin{pmatrix} \overline{d^b} \\ \overline{u^b} \end{pmatrix}_R \end{pmatrix} \tag{3}$$

Formally in terms of the underlying fields, the construction takes

$$\begin{array}{l} \psi = \frac{1}{2}(1-\gamma_5)\psi_{(\nu,e^-)} + \frac{1}{2}(1+\gamma_5)\psi_{(\overline{d^g},\overline{u^g})} \\ \chi = \frac{1}{2}(1-\gamma_5)\psi_{(u^r,d^r)} + \frac{1}{2}(1+\gamma_5)\psi_{(\overline{d^b},\overline{u^b})} \end{array} \tag{4}$$

From the conventional point of view these fields have rather peculiar quantum numbers. For example, the left and right parts have different electric charges. Electromagnetism now violates parity. The left and right parts also have different strong quantum numbers; the strong interactions violate parity as well. Finally, the components have different masses; parity is violated in the

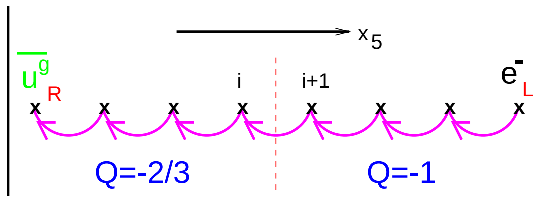

Figure 5: Pairing the electron with the anti-green-up-quark.

Higgs mechanism. Making the $SU(2)$ vector-like forces parity violation into all the other interactions.

The different helicities of these fields also have variant baryon number. This is directly related to the known baryon violating processes through weak "instantons" and axial anomalies.[3] As discussed earlier, the axial anomaly arises from a level flow out of the Dirac sea.[2] This generates a spin flip in the fields, *i.e.* $e_L^- \to (\overline{u^g})_R$. Because of my peculiar particle identification, this does not conserve charge, with $\Delta Q = -\frac{2}{3} + 1 = \frac{1}{3}$. This would be a disaster for electromagnetism were it not for the other Dirac doublet simultaneously flipping, *i.e.* $d^r{}_L \to (\overline{u^b})_R$, with a compensating $\Delta Q = -\frac{1}{3}$. This is anomaly cancelation, with the total $\Delta Q = \frac{1}{3} - \frac{1}{3} = 0$. Only when both doublets are considered together is the $U(1)$ symmetry restored. In the overall process baryon number remains violated, with $L + Q \to \overline{Q} + \overline{Q}$. This is the famous " 't Hooft vertex."[3]

This discussion has been in the continuum. Now I return to the lattice, and use the Kaplan-Shamir approach for fermions.[13 14 15] In this picture, our four dimensional world is a "4-brane" embedded in 5-dimensions. The complete lattice is a five dimensional box with open boundaries, and the parameters are chosen so the physical quarks and leptons appear as surface zero modes.

I now insert the above pairing into this five dimensional scheme. In particular, I consider the left handed electron as a zero mode on one wall and the right-handed anti-green-up quark as the partner mode on the other wall, as sketched in Fig. 5. This provides a lattice regularization for the $SU(2)$ of the weak interactions.

However, since these two particles have different electric charge, $U(1)_{EM}$ must be broken in the interior of the extra dimension. I now proceed in analogy to the "waveguide" picture[23] and restrict this charge violation to ΔQ to one layer at some interior $x_5 = i$. Then the fermion hopping term from $x_5 = i$ to $i + 1$

$$\overline{\psi}_i P \psi_{i+1} \qquad (P = \gamma_5 + r) \tag{5}$$

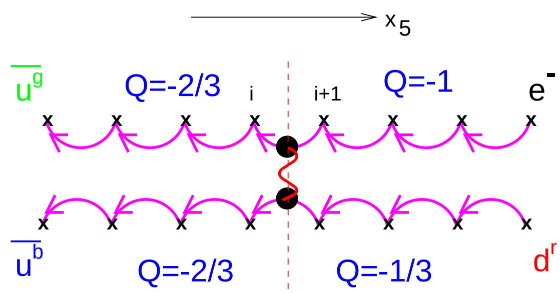

Figure 6: Transferring charge between the doublets introduces a four-fermion coupling.

is a $Q = 1/3$ operator. At this layer, electric charge is not conserved. This is unacceptable and needs to be fixed.

To restore the $U(1)$ symmetry I must transfer the charge from ψ to the compensating doublet χ. For this I replace the sum of hoppings with a product on the offending layer

$$\overline{\psi}_i P\psi_{i+1} + \overline{\chi}_i P\chi_{i+1} \longrightarrow \overline{\psi}_i P\psi_{i+1} \times \overline{\chi}_i P\chi_{i+1} \tag{6}$$

This introduces an electrically neutral four fermi operator. It is explicitly baryon violating, involving a "lepto-quark/diquark" exchange, as sketched in Fig. 6. One might think of the operator as representing a "filter" at $x_5 = i$ through which only charge compensating pairs of fermions can pass.

In five dimensions there is no chiral symmetry. Even for the free theory, combinations like $\overline{\psi}_i P\psi_{i+1}$ have vacuum expectation values. I use such as a "tadpole," with χ generating an effective hopping for ψ and *vice versa*.

Actually the above four fermion operator is not quite sufficient for all chiral anomalies, which can also involve right handed singlet fermions. To correct this I need explicitly include the right handed sector, adding similar four fermion couplings (also electrically neutral).

Having fixed the $U(1)$ of electromagnetism, I restore the strong $SU(3)$ with an antisymmetrization $Q^r Q^g Q^b \longrightarrow \epsilon^{\alpha\beta\gamma} Q^\alpha Q^\beta Q^\gamma$. Although the quarks reside at different locations in the fifth dimension, this is irrelevant since the $SU(3)$ symmetry need only be local in four-dimensional space-time. As for the electromagnetic case, additional left-right inter-sector couplings are needed to correctly obtain the effects of topologically non-trivial strong gauge fields. These are of the same form as the strong 't Hooft vertex.

An alternative view folds the lattice about the interior of the fifth dimension, placing all light modes on one wall and having the multi-fermion operator

on the other. This is the model of Ref. [21], with the additional inter-sector couplings correcting a technical error.[24]

Unfortunately the scheme is still non rigorous. The most serious worry is that the four fermion coupling might induce an unwanted spontaneous symmetry breaking of one of the gauge symmetries. I need a strongly coupled paramagnetic phase without spontaneous symmetry breaking.[25] Ref. [21] showed how strongly coupled zero modes preserve the desired symmetries, but the analysis ignored contributions from heavy modes in the fifth dimension.

Assuming all works as desired, the model raises several interesting questions. A variation using a Majorana mass term on the extra wall seems quite promising for formulating supersymmetric Yang-Mills theory on the lattice.[26] Can a related scheme give a natural formulation for more general supersymmetric theories? Above I needed a right handed neutrino to provide all quarks with partners. Is there some variation to avoids this particle, which completely decouples in the continuum limit? Another question concerns numerical simulations; is the effective action positive? Finally, I have used details of the usual standard model, leaving open the question of whether this model is somehow special. Can I always use multi-fermion couplings to eliminate undesired modes in other anomaly free chiral theories? There is much more to do!

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